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On the development of the wake behind the trailing edge of a flat plate

By S. H. SMITH
Department of Mathematics University of Toronto, Toronto 5, Canada.

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A stream with constant velocity U is impulsively started at time $t = 0$ past the trailing edge of a semi-infinite flat plate. According to boundary-layer theory, it is found that the flow at a distance x downstream from the trailing edge is unaware of the presence of the plate when $x > Ut$; at time $t = x/U$ there is then a discontinuity in the velocity normal to the plate. It is the neglect of diffusion parallel to the axis of the plate that introduces the discontinuity, and when the complete Navier–Stokes equations are considered for $t \simeq x/U$, a solution is found that can be matched with that gained from boundary-layer arguments.

1. Introduction

Some time ago Stewartson (1951) considered the following problem: a semi-infinite flat plate is at rest in a slightly viscous liquid when, at time $t = 0$, a uniform stream of constant velocity U is impulsively set up past the leading edge of the plate. This was tackled as a boundary-layer problem, so that for times t less than x/U, at points a distance x downstream from the leading edge, the Rayleigh solution for the flow past an infinite plate represents the motion. At later times, however, the steady Blasius solution for the semi-infinite plate eventually dominates. The manner in which the motion passes from one limiting case to the other has been the cause of certain controversy recently. Stewartson himself indicated that the effect of the leading edge is passed by convection with velocity U along the edge of the boundary layer, arriving a distance x downstream when $\tau = Ut/x = 1$; diffusion then transmits this effect through the boundary layer to the plate. Mathematically, an essential singularity is expected at $\tau = 1$. No formal proof could be given, but a solution of the equations was found that did possess such a singularity. The analysis in this paper was generalized by Smith (1967) for the equivalent flow past a wedge, and a similar, tentative conclusion followed.

In a recent paper by Tokuda (1968), the results of Stewartson were disputed, though Stewartson & Brown (see corrigendum to Tokuda 1968) rightly observe that his conclusions were based on inaccurate numerical data, and that his proof of the existence of a series solution for the velocity is false.

To the present author at least, the search for a solution with an essential singularity at $\tau = 1$ seems the only one likely to succeed *when the equations considered are the boundary-layer equations.* The boundary-layer equations for unsteady flows neglect the diffusion of vorticity parallel to the stream; they include only diffusion perpendicular, and convection parallel to the mainstream flow. Convection is governed by hyperbolic partial differential equations which preserve discontinuities, and diffusion by parabolic partial differential equations which 'smooth out' discontinuities immediately—mathematically by the presence of essential singularities. The author is aware of the numerical solution by Hall (1968) of the boundary-layer equations for the Stewartson problem; a smooth joining of the two limiting cases is exhibited. Mathematically, the discontinuity is a consequence of solving the linearized boundary-layer equations; it is then found that the introduction of the non-linear terms ensures the existence of a smooth solution with the essential singularity.

A factor neglected by Stewartson is the influence of diffusion acting parallel to the stream. The full Navier–Stokes equations would have to be considered if this effect were included, but the mathematics involved is very difficult and has not been attempted here. However, any solution would show that the knowledge of the leading edge is transmitted immediately throughout the flow field; here the only discontinuity would be at the initial moment of time.

In the present work, therefore, we consider the same physical situation except that the uniform stream flows in the opposite direction; that is, we take the edge of the semi-infinite plate to be a trailing edge. With this change the mathematics becomes more amenable to solution and the main features of the flow are displayed. After stating the problem in §2, we first consider the solution of the boundary-layer equations when the variation from a uniform stream is small; this enables us to linearize the equations. For points in the wake region, when $0 < \tau = Ut/x < 1$, the flow has a constant velocity U parallel to the plate. When $\tau = 1$, the influence of the trailing edge is first noticed with a discontinuity in the velocity normal to the axis of the plate.

A discussion follows of the nature of the flow as $\tau \to \infty$. and it is found that the approach to the limiting solution is by means of an exponential decay. When a precise asymptotic analysis is carried through, it is interesting to observe that it is the process of convection acting within the boundary layer that transmits the effect of the trailing edge through the wake.

In an attempt to eliminate the singularity at $\tau = 1$, the Navier–Stokes equations, linearized in the same manner as the boundary-layer equations, are investigated in the region $\tau \simeq 1$ as a singular perturbation problem. A solution is found which matches with the behaviour in the boundary-layer solution for both $0 < 1-\tau \ll 1$ and $0 < \tau-1 \ll 1$. The smooth joining that was anticipated on physical grounds is therefore proved. When the complete (non-linear) Navier–Stokes equations are investigated, it is found that the linearization procedure is certainly valid for regions at the edge of the boundary layer. As points closer to the axis within the boundary layer are considered, the non-linear terms become more important. However, it is unusual to note that when linear and non-linear terms are of equal importance, the leading term in the solution of the differential equation is just that found from the linear Navier–Stokes equations; the higher terms do differ though. This fact considerably extends the validity of the linear solution.

It is of some interest to consider the effect of the non-linear terms in the boundary-layer equations. In the work of Stewartson (1951) it was the influence of these terms that allowed the discontinuity from the linearized equations to be smoothed out by the essential singularity. In § 6, a summary is given of arguments which indicate that in the trailing edge situation the discontinuity is not removed by the non-linear terms. There is no essential singularity at $\tau = 1$; in fact the dominant term of the solution of the linear boundary-layer equations near $\tau = 1$ is also the dominant term from the non-linear equations. The physical explanation offered is as follows: in the flow past the leading edge the effect of the edge is convected downstream at a velocity less than that of the uniform stream. At the edge of the boundary layer this difference is certainly very small, but it is non-zero. The presence of the plate instantaneously retards the flow at all points downstream of the leading edge at the initial time. In contrast, the vorticity downstream from the trailing edge is zero at the initial time, and the effect of the edge is convected at exactly the free-stream velocity. Initially the vorticity is discontinuous along the line $x = 0$, and so this line of discontinuity moves downstream with the constant velocity U when the effect of diffusion parallel to the plate is neglected. That is, the discontinuity is preserved at the value $\tau = 1$, and it can only be removed through considering the Navier-Stokes equations.

Finally, we consider the limitations of the model of a semi-infinite flat plate to describe realistic flows.

2. Statement of the problem

We consider the problem as one with a constant velocity U impulsively set up in the main stream at time $t = 0$, while the plate remains at rest. The motion is two dimensional, so we take the origin of the co-ordinate system as the trailing edge of the flat plate which otherwise occupies the negative part of the x axis. If u and v are the components of velocity parallel to the x and y axes respectively, then the Navier–Stokes equations are

$$u_x + v_y = 0, \tag{2.1}$$

$$u_t + uu_x + vu_y = -\rho^{-1}p_x + \nu\nabla^2 u, \tag{2.2}$$

$$v_t + uv_x + vv_y = -\rho^{-1}p_y + \nu\nabla^2 v; \tag{2.3}$$

$p(x, y, t)$ is the pressure and ρ, ν are the constants representing the density and kinematic viscosity of the fluid.

When a stream function $\psi(x, y, t)$ is defined by $u = \psi_y, v = -\psi_x$ the equation of continuity (2.1) is immediately satisfied. The pressure p can then be eliminated from the momentum equations (2.2), (2.3) for

$$\omega_t - \frac{\partial(\psi, \omega)}{\partial(x, y)} = \nu\nabla^2\omega, \tag{2.4}$$

when $\omega = \nabla^2\psi$; this is the Helmholtz equation for the vorticity $\omega(x, y, t)$. Because of a symmetry about the x axis, the solution of these equations is considered for $y \geqslant 0$ only.

The boundary and initial conditions can be stated as follows:

$$u = v = 0 \quad \text{when} \quad y = 0, \quad x < 0, \quad t \geqslant 0; \tag{2.5a}$$

$$u_y = v = 0 \quad \text{when} \quad y = 0, \quad x > 0, \quad t \geqslant 0; \tag{2.5b}$$

$$u = U, \quad v = 0 \quad \text{when} \quad y > 0, \quad x > 0, \quad t = 0; \tag{2.5c}$$

$$u \simeq U, \quad v \simeq 0 \quad \text{when} \quad y \to \infty, \quad t \geqslant 0; \tag{2.5d}$$

$$u \simeq U, \quad v \simeq 0 \quad \text{when} \quad x \to +\infty, \quad t \geqslant 0. \tag{2.5e}$$

The final condition to be stated is that for $x \to -\infty$. Here there is no knowledge of the trailing edge so that the velocities are those for the flow past an infinite plate. That is,

$$u \simeq U \operatorname{erf}\left(\frac{y}{2(\nu t)^{\frac{1}{2}}}\right), \quad v \simeq 0 \quad \text{as} \quad x \to -\infty, \quad t \geqslant 0, \tag{2.5f}$$

where

$$\operatorname{erf} z = 2\pi^{-\frac{1}{2}} \int_0^z e^{-u^2} du;$$

this was given by Rayleigh (1911). The only other point noted here is that, eventually, the velocities tend to zero throughout the flow field. However, the manner of this decay does not interest us; it is the somewhat artificial result of taking a semi-infinite plate rather than one of finite length.

The conditions to be satisfied have been stated for the set of equations (2.1)–(2.3); it is a straightforward matter to adjust these for the equation (2.4).

3. A boundary-layer solution

It is the assumption of the Prandtl boundary-layer theory for the unsteady flow past a flat plate that the motion is represented by a balance between convection parallel to, and diffusion normal to, the axis of the plate; the pressure gradient is zero. This leads to the equations (cf. Rosenhead 1963),

$$u_x + v_y = 0, \tag{3.1}$$

$$u_t + uu_x + vu_y = \nu u_{yy}, \tag{3.2}$$

from (2.1), (2.2). In this approximation we neglect the action of diffusion parallel to the plate; convection alone acts in the positive x direction, so that (2.5f) must represent the velocities for all $x < 0$. In the remainder of this section we consider $x \geqslant 0$ alone, and set the condition

$$u = U \operatorname{erf}\left(\frac{y}{2(\nu t)^{\frac{1}{2}}}\right), \quad v = 0 \quad \text{when} \quad x = 0, \quad t \geqslant 0, \quad y > 0 \tag{3.3}$$

to replace (2.5a, f); (2.5b–e) remain. These non-linear boundary-layer equations cannot, of course, be solved completely; further assumptions need now be made to gain the information required.

Here our interest centres on the development of the wake downstream from the trailing edge. To begin, therefore, we consider the situation when the motion differs only slightly from the basic flow of a uniform stream; that is, we neglect the products of v and u-U in the boundary-layer equation (3.2) to give the linear differential equation

$$u_t + Uu_x = \nu u_{yy} \quad (x, y, t \geqslant 0). \tag{3.4}$$

The solution of this equation, together with the conditions (2.5*b*–*e*), (3.3), is considered in an attempt to describe the flow at a time soon after the disturbance due to the plate reaches the point $P(x, y)$ in the wake region, particularly for P at the edge of the boundary layer. The neglect of the non-linear terms certainly does raise important points; however, we delay a full discussion until a later section. The equation (3.4) is equivalent to that derived by Stewartson (1951); a similar method of solution to the one he used is adopted here.

We define

$$\tau = Ut/x, \quad \zeta = y/(\nu t)^{\frac{1}{2}}, \tag{3.5}$$

and from dimensional arguments it is clear that u is a function of ζ and τ only. Consequently, $u(\zeta, \tau)$ satisfies

$$u_{\zeta\zeta} + \tfrac{1}{2}\zeta u_\zeta + \tau(\tau-1)\, u_\tau = 0 \tag{3.6}$$

subject to the conditions $u_\zeta = 0$ on $\zeta = 0$; $u \to U$ as both $\zeta \to \infty$ and $\tau \to 0$; $u \simeq U \operatorname{erf} \frac{1}{2}\zeta$ as $\tau \to \infty$, $\zeta \neq 0$. A solution is sought in the form of a Fourier cosine transform, defining

$$\bar{u}(\alpha, \tau) = \int_0^\infty u(\zeta, \tau) \cos \alpha\zeta \, d\zeta.$$

With $(u_\zeta)_{\zeta=0} = 0$, the transform of (3.6) becomes the first-order partial differential equation

$$\tfrac{1}{2}\alpha\bar{u}_\alpha - \tau(\tau-1)\bar{u}_\tau + (\alpha^2 + \tfrac{1}{2})\bar{u} = 0,$$

which has the solution

$$\bar{u} = \alpha^{-1} e^{-\alpha^2} G\{\alpha^2\tau^{-1}(\tau-1)\},$$

where G is an arbitrary function. That is,

$$u(\zeta, \tau) = \int_0^\infty \frac{e^{-\alpha^2}}{\alpha} G\left\{\frac{\alpha^2}{\tau}(\tau-1)\right\} \cos \zeta\alpha \, d\alpha, \tag{3.7}$$

when the constant for the inverse transform is absorbed into G. From the condition $\tau \to \infty$ we find

$$\int_0^\infty \alpha^{-1} e^{-\alpha^2} G(\alpha^2) \cos \zeta\alpha \, d\alpha = U \operatorname{erf} \tfrac{1}{2}\zeta.$$

Taking the inverse transform (Erdelyi *et al.* 1954, p. 73),

$$\alpha^{-1} e^{-\alpha^2} G(+\alpha^2) = U\delta(\alpha) - 4\pi^{-\frac{3}{2}} U e^{-\alpha^2} \Phi(\tfrac{1}{2}; \tfrac{3}{2}; \alpha^2),$$

where $\Phi(a; c; z)$ is the Humbert notation for the confluent hypergeometric function (Erdelyi *et al.* 1953, p. 248) and $\delta(\alpha)$ is the Dirac delta function. The other conditions are already satisfied unless $\tau \to 0$ and $\zeta \to \infty$ simultaneously such that $\tau\zeta^2$ is constant. In this case we require

$$\alpha^{-1} G(-\alpha^2) = U\delta(\alpha).$$

Therefore, if $\tau < 1$, the solution is $u = U$, though for $\tau \geqslant 1$ we have, after some simplification,

$$u = U - \frac{4U}{\pi^{\frac{3}{2}}}\left(\frac{\tau-1}{\tau}\right)^{\frac{1}{2}} \int_0^\infty e^{-\alpha^2} \cos \zeta\alpha \, \Phi\left\{\tfrac{1}{2}; \tfrac{3}{2}; \frac{\alpha^2}{\tau}(\tau-1)\right\} d\alpha. \tag{3.8}$$

When $0 \leqslant \tau - 1 \ll 1$,

$$u - U \simeq -2\pi^{-1} U(\tau-1)^{\frac{1}{2}} e^{-\frac{1}{4}\zeta^2}; \tag{3.9}$$

more generally, we can expand the integral of (3.7) into the infinite series

$$u = U - \frac{2U}{\pi}\left(\frac{\tau-1}{\tau}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+1)}\left(\frac{\tau-1}{\tau}\right)^n (e^{-\frac{1}{4}\zeta^2})^{(2n)}, \tag{3.10}$$

when the $(2n)$ superscript represents the $2n$th derivative.

These results can be interpreted immediately: the velocity $u(x, y, t)$ is constant at the point $P(x, y)$ until a time $t = x/U$ has elapsed. Accordingly, there is a finite time within which the flow in the wake is unaware of the presence of the plate because the disturbance due to the trailing edge is transmitted through the liquid by convection at the mainstream velocity. At $\tau = 1$, the velocity u, and all its derivatives with respect to y are continuous. However, u_x is discontinuous and so, from the equation of continuity, v is also discontinuous. This conclusion is physically unrealistic, and can be taken to be a natural consequence of neglecting the derivatives with respect to x in the boundary-layer approximation. Alternatively, it can be argued that the discontinuity present in the solution of the linear equation is removed when the non-linear terms are included, and that the real flow is more accurately described in this way. Stewartson (1951) followed the second line of reasoning when he considered the flow past the leading edge. These two possibilities are closely investigated in the following sections.

According to the linear boundary-layer approximation, the vorticity ω is given by u_y. From (3.9) this then indicates $\omega = 0$ for $\tau < 1$, and

$$\omega = \frac{U}{\pi(\nu t)^{\frac{1}{2}}}\left(\frac{\tau-1}{\tau}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+1)}\left(\frac{\tau-1}{\tau}\right)^n (\zeta e^{-\frac{1}{4}\zeta^2})^{(2n)} \tag{3.11}$$

for $\tau \geqslant 1$.

After the work described in this paper had been completed, it was found possible to sum this series. The terms in (3.11) can be rearranged to give an infinite series with terms in $(\tau - 1)$ rather than $(\tau-1)/\tau$. The resulting expression is just

$$\omega = \frac{U}{\pi(\nu t)^{\frac{1}{2}}} e^{-\frac{1}{4}\zeta^2} \sum_{n=0}^{\infty} \frac{(-1)^n \zeta^{2n+1}}{2^{2n} n!(2n+1)} (\tau-1)^{n+\frac{1}{2}} \quad (\tau \geqslant 1),$$

which is the series expansion for the function

$$\omega = \frac{U}{(\pi\nu t)^{\frac{1}{2}}} e^{-\frac{1}{4}\zeta^2} \operatorname{erf}\left(\frac{\zeta(\tau-1)^{\frac{1}{2}}}{2}\right) \quad (\tau \geqslant 1). \tag{3.12}$$

This rearrangement is a purely formal procedure; however, it is now easily seen that (3.12) does in fact satisfy both the differential equation and boundary conditions required, and so represents the solution for the problem. Corresponding to (3.12) we can gain

$$u = U - \frac{U}{\pi^{\frac{1}{2}}}\int_{\zeta}^{\infty} e^{-\frac{1}{4}\rho^2} \operatorname{erf}\left(\frac{\rho(\tau-1)^{\frac{1}{2}}}{2}\right) d\rho \quad (\tau \geqslant 1),$$

from (3.10).

When the asymptotic expansion is taken for (3.12), we have

$$\omega \simeq U(\pi\nu t)^{-\frac{1}{2}}\{e^{-\frac{1}{4}\zeta^2} - 2(\pi\zeta^2\tau)^{-\frac{1}{2}} e^{-\frac{1}{4}\zeta^2\tau}\} \quad \text{for} \quad \tau \to \infty, \quad \zeta = O(1).$$

The second term is essentially the correction due to the disturbance of the trailing edge at the edge of the boundary layer for large times t; the variable

$\zeta^2\tau = Uy^2/\nu x$ is time independent. It is seen, therefore, that the disturbance is carried away by convection, and particularly that it is concentrated near ζ small, i.e. closer to the axis. It is indeed interesting to observe the role of convection here in transmitting the effect of the trailing edge within the boundary layer itself, and the author is grateful to a referee for bringing this point to his attention.

4. A solution in the neighbourhood of $\tau = 1$

It is convenient to consider the Navier–Stokes equations in the form (2.4). Again we begin by considering the flow when it differs slightly from that of a uniform stream, so that it is possible to write $\psi = Uy$ in the Jacobian of (2.4) to give

$$\omega_t + U\omega_x = \nu\nabla^2\omega. \tag{4.1}$$

These circumstances are the same as those under which (3.4) was considered in the previous section; now, however, the ω_{xx} term is included to represent diffusion parallel to the x axis.

Together with the non-dimensional variables ζ and τ (given in (3.5)), we further define

$$\eta = \frac{x}{(\nu t)^{\frac{1}{2}}}.$$

When the function $H(\zeta, \eta, \tau)$ is introduced by $\omega = U(\nu t)^{-\frac{1}{2}}H$, it is seen that H satisfies the linear partial differential equation

$$H_{\zeta\zeta} + \tfrac{1}{2}\zeta H_\zeta + \tfrac{1}{2}H + \tau(\tau-1)H_\tau = \tau\eta H_\eta - \tfrac{1}{2}\eta H_\eta - H_{\eta\eta} - 2\tau\eta^{-2}H_\tau - \tau^2\eta^{-2}H_{\tau\tau} \tag{4.2}$$

from (4.1). When $\eta \to \infty$, and $\partial/\partial\eta = 0$, the right-hand side of (4.2) is zero, so that the resultant equation for the vorticity in terms of H as a function of ζ and τ is equivalent to the boundary-layer equation (3.6) for the velocity $u(\zeta, \tau)$. After setting appropriate conditions the solution would then be given by (3.11).

Generally, it is known that the singular points of a differential equation occur where the coefficient of a highest order derivative is equal to zero. Now when η is infinite and $\tau = 1$, the coefficients of both the $H_{\tau\tau}$ and H_τ terms in (4.2) are zero. In the region under consideration it is necessary that the coefficients of these terms are of finite order, together with the coefficients of the other highest derivatives. Physically, this ensures that the processes of diffusion (in both directions) and convection are in balance.

We therefore introduce the transformation

$$\sigma = (\tau - 1)\eta \tag{4.3}$$

to replace τ; (4.2) then becomes

$$H_{\zeta\zeta} + \tfrac{1}{2}\zeta H_\zeta + \tfrac{1}{2}H - (\sigma+\eta)H_\eta + \tfrac{1}{2}\sigma H_\sigma + 2\eta^{-2}(\sigma+\eta)H_\sigma + \tfrac{1}{2}\eta H_\eta + 2\sigma\eta^{-1}H_{\sigma\eta} + H_{\eta\eta} + \eta^{-2}\{(\sigma+\eta)^2+\sigma^2\}H_{\sigma\sigma} = 0. \tag{4.4}$$

When $\eta \gg 1$ and $|\tau - 1| \ll 1$ such that $\sigma = O(1)$, the coefficients of $H_{\sigma\sigma}$ and H_σ are both $O(1)$. The transformation (4.3) is a stretching transformation in the terminology of singular perturbation problems.

The boundary conditions are now set in terms of ζ and σ because a solution is sought for (4.4) for large η. At $\zeta = 0$ we require $H = 0$, since the vorticity is zero on $y = 0$, $x > 0$, cf. (2.5*b*). Further, $H \to 0$ as $\zeta \to \infty$ and also as $\sigma \to -\infty$. Finally we match H onto the dominant term

$$\pi^{-1}(\tau-1)^{\frac{1}{2}}\zeta e^{-\frac{1}{4}\zeta^2} \tag{4.5}$$

of (3.10) as $\sigma \to +\infty$.

For the first step in the solution it is possible for us to write

$$H = \zeta e^{-\frac{1}{4}\zeta^2} M(\sigma, \eta) \tag{4.6}$$

for some function M; in this way the conditions for ζ can be satisfied as well as (4.5). The corresponding differential equation for M becomes

$$(1+2\sigma\eta^{-1}+2\sigma^2\eta^{-2})\,M_{\sigma\sigma}+M_{\eta\eta}+2\sigma\eta^{-1}M_{\sigma\eta} + (\tfrac{1}{2}\sigma+2\eta^{-1}+2\sigma\eta^{-2})\,M_{\sigma}-(\sigma+\tfrac{1}{2}\eta)\,M_{\eta}+\tfrac{1}{2}M = 0,$$

the solution of which must match with $\pi^{-1}\sigma^{\frac{1}{2}}\eta^{-\frac{1}{2}}$ from (4.5). Therefore we write $M = \eta^{-\frac{1}{2}}m(\sigma)$, and then note, on retaining only the dominant terms for large η, that

$$m''+\tfrac{1}{2}\sigma m'-\tfrac{1}{4}m = 0, \tag{4.7}$$

where dashes denote differentiation with respect to σ. The general solution of this ordinary differential equation is

$$m(\sigma) = A\Phi(-\tfrac{1}{4};\ \tfrac{1}{2};\ -\tfrac{1}{4}\sigma^2)+B(-\tfrac{1}{4}\sigma^2)^{\frac{1}{2}}\Phi(\tfrac{1}{4};\ \tfrac{3}{2};\ -\tfrac{1}{4}\sigma^2), \tag{4.8}$$

where A and B are constants (possibly complex). As a function of the complex variable z, $\Phi(a;c;z)$ is defined in the z plane cut along the negative real axis; hence $m(\sigma)$ has different representations for $\sigma > 0$ and $\sigma < 0$ while still retaining continuous derivatives of all orders at $\sigma = 0$. We require m to have an exponential decay as $\sigma \to -\infty$, and $m \sim \pi^{-1}\sigma^{\frac{1}{2}}$ as $\sigma \to +\infty$; the asymptotic expansions for Φ (Erdelyi *et al.* 1953, p. 278) show

$$A = 2^{-\frac{1}{2}}\pi^{-\frac{3}{2}}\,\Gamma(\tfrac{3}{4}) \quad\text{and}\quad B = -i(2\pi)^{-\frac{3}{2}}\,\Gamma(\tfrac{1}{4}). \tag{4.9}$$

We note, in particular, that the values (4.9) imply

$$m \sim 2^{-\frac{1}{2}}\pi^{-1}(-\sigma)^{-\frac{3}{2}}e^{-\frac{1}{4}\sigma^2} \quad\text{as}\quad \sigma \to -\infty. \tag{4.10}$$

Collecting these results together, we can finally write

$$\omega \sim \frac{U}{(\nu t)^{\frac{1}{2}}}\frac{\zeta e^{-\frac{1}{4}(\zeta^2+\sigma^2)}}{(2\pi)^{\frac{3}{2}}\eta^{\frac{1}{2}}}\{2\Gamma(\tfrac{3}{4})\,\Phi(\tfrac{3}{4};\tfrac{1}{2};\tfrac{1}{4}\sigma^2)\pm\tfrac{1}{2}\Gamma(\tfrac{1}{4})\sigma\,\Phi(\tfrac{5}{4};\tfrac{3}{2};\tfrac{1}{4}\sigma^2)\} \tag{4.11}$$

for $\sigma \gtrless 0$ as the vorticity when $\eta \gg 1$, $|\tau-1| \ll 1$.

The result (4.11) clearly indicates a process whereby the effect of the plate is initially, though only slightly noticed at a point in the wake through the process of diffusion; its effect is then rapidly increased when $\tau \sim 1$ as convection comes to dominate the motion. As τ increases in value, the boundary-layer solution (3.11) will give an accurate representation for the velocity with an error of the order of $e^{-\frac{1}{4}\eta^2}$ as long as the assumptions u–U, $v \ll U$ are valid. This will certainly be true at the edge of the boundary layer, though at points well within this layer the full non-linearity of the differential equations will have to be faced.

We just note here that the matching can be continued for higher terms. When we write $H = (\zeta e^{-\frac{1}{4}\zeta^2})^{(2n)}\, \eta^{-(n+\frac{1}{2})} m_n(\sigma)$, and substitute into (4.4), the dominant terms for $\eta \gg 1$ lead to an ordinary differential equation for $m_n(\sigma)$ with solution

$$m_n = A_n \Phi(-\tfrac{1}{4} - \tfrac{1}{2}n;\, \tfrac{1}{2};\, -\tfrac{1}{4}\sigma^2) + B_n(-\tfrac{1}{4}\sigma^2)^{\frac{1}{2}}\, \Phi(\tfrac{1}{4} - \tfrac{1}{2}n;\, \tfrac{3}{2};\, -\tfrac{1}{4}\sigma^2)$$

for constants A_n and B_n; this generalizes (4.8). The constants are calculated on satisfying the conditions $m_n \to 0$ when $\sigma \to -\infty$; m_n is proportional to $\sigma^{n+\frac{1}{2}}$ when $\sigma \to +\infty$. The details are not completed here.

The main question to consider at this juncture in the work is the validity of the linearization procedure that resulted in (4.1). With this end in view, the complete Navier–Stokes equations are considered in terms of the independent variables ζ, η, σ. The function $F(\zeta, \eta, \sigma)$ is defined from the stream function ψ by

$$\psi = U(\nu t)^{\frac{1}{2}}(\zeta + F);$$

this isolates the part due to the uniform stream. The vorticity equation (2.4) is then

$$\begin{aligned} H_{\zeta\zeta} + \tfrac{1}{2}\zeta H_\zeta + \tfrac{1}{2}H + H_{\eta\eta} - \tfrac{1}{2}\eta H_\eta - \sigma H_\eta + 2\sigma\eta^{-1}H_{\sigma\eta} + (1 + 2\sigma\eta^{-1} + 2\sigma^2\eta^{-2})H_{\sigma\sigma} \\ + \tfrac{1}{2}\sigma H_\sigma + 2(\eta^{-1} + \sigma\eta^{-2})H_\sigma = (\sigma + \eta)\{(F_\sigma H_\zeta - F_\zeta H_\sigma) + (F_\zeta H_\eta - F_\eta H_\zeta)\}, \end{aligned} \tag{4.12}$$

where

$$H = F_{\zeta\zeta} + F_{\eta\eta} + 2\sigma\eta^{-1}F_{\sigma\eta} + 2(\eta^{-1} + \sigma\eta^{-2})F_\sigma + (1 + 2\sigma\eta^{-1} + 2\sigma^2\eta^{-2})F_{\sigma\sigma}. \tag{4.13}$$

From the linear analysis we have found that

$$H \sim \eta^{-\frac{1}{2}}\zeta e^{-\frac{1}{4}\zeta^2} m(\sigma) \quad \text{for} \quad \zeta \gg 1, \quad \eta \gg 1, \quad \sigma = O(1); \tag{4.14}$$

consequently,

$$F \sim 4\eta^{-\frac{1}{2}}\zeta^{-1}e^{-\frac{1}{4}\zeta^2} m(\sigma) \quad \text{for} \quad \zeta \gg 1, \quad \eta \gg 1, \quad \sigma = O(1). \tag{4.15}$$

When these asymptotic representations are substituted into (4.12), it is observed that the linear terms are of the order $\eta^{-\frac{1}{2}}\zeta^3 e^{-\frac{1}{4}\zeta^2}$, whereas the non-linear terms are of the order $\zeta e^{-\frac{1}{2}\zeta^2}$. Therefore the neglect of the non-linear terms on the right-hand side is justified when

$$\zeta^2 e^{\frac{1}{4}\zeta^2} \gg \eta^{\frac{1}{2}}. \tag{4.16}$$

The variable $\sigma = (\tau - 1)\eta$ is finite, so that $\eta \to \infty$ as $\tau \to 1$; (4.16) shows that ζ need tend to infinity no quicker than $(2 \log \eta)^{\frac{1}{2}}$ as $\tau \to 1$. This indicates that there does exist a region downstream from the origin at the edge of the boundary layer where the linearized Navier–Stokes equations are sufficient to describe the real flow as $\tau \to 1$. At points further into the boundary layer the non-linear terms must be taken into account.

We now introduce new independent variables

$$\phi = \eta^{-1}\zeta^4 e^{\frac{1}{2}\zeta^2} \quad \text{and} \quad \chi = \eta^{-1}\zeta$$

to replace ζ and η. The variable ϕ is taken to be $O(1)$, which requires $\chi \ll 1$ when ζ is large; we maintain $\sigma = O(1)$ as before. The asymptotic condition (4.15) becomes $F \sim 4\chi\phi^{-\frac{1}{2}}m(\sigma)$ as $\phi \to \infty$; this enables us to write

$$F = \chi f(\phi, \sigma), \tag{4.17}$$

which is substituted into the equations (4.12), (4.13). It is expected that (4.17) represents the leading term for χ small in the expression for the stream function within the required region. When the dominant terms only are retained, the resultant partial differential equation for f is

$$\phi^2 f_{\phi\phi\phi} + \tfrac{7}{2}\phi f_{\phi\phi} + \tfrac{3}{2} f_\phi + \phi f_\phi f_{\phi\sigma} - \phi f_\sigma f_{\phi\phi} - f_\sigma f_\phi = 0; \tag{4.18}$$

the ratio of terms neglected to those retained is $O(\zeta^{-2})$. The boundary conditions to be posed are $f \sim 4\phi^{-\frac{1}{2}} m(\sigma)$ as $\phi \to \infty$, $f \sim 4\pi^{-1}\phi^{-\frac{1}{2}}\sigma^{\frac{1}{2}}$ as $\sigma \to \infty$ and

$$f \sim 2^{\frac{3}{2}}\pi^{-1}\phi^{-\frac{1}{2}}(-\sigma)^{-\frac{3}{2}} e^{-\frac{1}{4}\sigma^2} \text{ as } \sigma \to -\infty.$$

A solution that satisfies (4.18) and all these conditions is in fact

$$f = 4\phi^{-\frac{1}{2}} m(\sigma). \tag{4.19}$$

That is, the solution of the linear part of the differential equation also satisfies the non-linear part when equated to zero. This could, of course, have been noted immediately from observing that the dominant terms gained from substituting (4.15) into (4.12) do cancel. Nevertheless, the formal analysis pursued above is necessary for later observations.

The function (4.19) is a solution of (4.18), but because insufficient boundary conditions have been imposed we cannot be certain yet that it is the unique solution. No condition has been stated at $\phi = 0$ on the boundary of the domain $-\infty < \sigma < \infty$, $\sigma \geqslant 0$. Uniqueness can be investigated by taking (4.19) to be the first term in an asymptotic series for ϕ large of the exact solution of (4.18). We write $f = 4\phi^{-\frac{1}{2}} m(\sigma) + f_1(\phi, \sigma)$, where $|f_1| \ll \phi^{-\frac{1}{2}}$ for $\phi \gg 1$, and f_1 does not upset the conditions as $\sigma \to \pm\infty$. When the quadratic terms in f_1 are rejected, the resultant linear differential equation is seen to be

$$(\phi^2 f_{1\phi\phi\phi} + \tfrac{7}{2}\phi f_{1\phi\phi} + \tfrac{3}{2} f_{1\phi}) - (4\phi^{\frac{1}{2}} m' f_{1\phi\phi} + 2\phi^{-\frac{1}{2}} m f_{1\phi\sigma} + 6\phi^{-\frac{1}{2}} m' f_{1\phi} + \phi^{-\frac{3}{2}} m f_{1\sigma}) = 0. \tag{4.20}$$

The terms in the first bracket dominate for $\phi \gg 1$, and when the other terms are neglected the differential equation can be formally integrated for

$$f_1 = A(\sigma)\phi^{-\frac{1}{2}} + B(\sigma)\log\phi + C(\sigma),$$

where A, B, C are arbitrary functions. To satisfy the condition $|f_1| \ll \phi^{-\frac{1}{2}}$ for $\phi \gg 1$ it is clear that the functions A, B and C are all identically zero.

The only other way in which a solution with continuous derivatives of all orders can arise from a linear equation such as (4.20) is through the presence of an essential singularity at some value $\phi = \phi_0 > 0$. Now essential singularities are only anticipated for values ϕ_0 which give a zero coefficient for the highest order derivative with respect to ϕ; it is immediately observed that there are no positive values ϕ_0 with this property in the present case. These arguments show that (4.19) is, in fact, the unique solution for all ϕ, σ in the given domain. The range of validity of the solutions (4.14), (4.15) is thereby increased; nevertheless the formulation of (4.18) involved neglecting terms that were $O(\zeta^{-2})$, so it is still not possible to take $\zeta = O(1)$.

We now consider the higher order terms. The approximation (4.17) is known to incur an error that is $O(\zeta^{-2})$. Now $\zeta^3 e^{\frac{1}{4}\zeta^2} = \phi\chi^{-1}$ and so for small χ we can write $\zeta^{-2} = -(2\log\chi)^{-1}$; the first two terms in the expansion for F are then given by

$$F = 4\chi\phi^{-\frac{1}{2}}m(\sigma) + \chi(\log\chi)^{-1}f^*(\phi,\sigma)$$

for some function f^*. The dominant terms gained when this is substituted into (4.12), (4.13) provide the linear differential equation

$$\phi^4 f^*_{\phi\phi\phi\phi} + \tfrac{13}{2}\phi^3 f^*_{\phi\phi\phi} + \tfrac{17}{2}\phi^2 f^*_{\phi\phi} + \tfrac{3}{2}\phi f^*_{\phi} - \phi^{-\frac{1}{2}} m'(4\phi^2 f^*_{\phi\phi\phi} + 12\phi f^*_{\phi\phi} + 3f^*_{\phi})$$
$$- \tfrac{1}{2}\phi^{-\frac{3}{2}} m(4\phi^2 f^*_{\phi\phi\sigma} + 4\phi f^*_{\phi\sigma} - f^*_{\sigma}) = 4\phi^{-1}(m'm'' - mm'''). \quad (4.21)$$

No general solution of (4.21) seems to be possible, though we can note that there exists the solution $f^* = A(\sigma)\phi^{-\frac{1}{2}}$, for all functions A, of the homogeneous differential equation. Particular solutions of the inhomogeneous equation for ϕ large and ϕ small are respectively

$$f^* = 8\phi^{-1}(m'm'' - mm''') \quad \text{and} \quad f^* = 2\phi^{\frac{1}{2}}\log\phi\, m^{-1}(mm'' - m'^2);$$

both are small in comparison with $\phi^{-\frac{1}{2}}$ in their separate domains. Consequently, the corrective effects for the dominant term (4.19) from the non-linear part of the differential equation do not enter the resultant expression for F until higher orders than the second. The details are not considered any further here.

5. Discussion

In § 3 the boundary-layer equations are linearized to give an understanding of the flow at the edge of the boundary layer; we now briefly consider the role of the non-linear terms in these equations in the neighbourhood of $\tau = 1$. Because the analysis is very long, in some places following closely that already given in § 4, the conclusions are just summarized here.

When the stream function $\psi(x,y,t)$ is written as $\psi = U(\nu t)^{\frac{1}{2}}\{\zeta + F(\zeta,\tau)\}$, F satisfies the differential equation

$$F_{\zeta\zeta\zeta} + \tfrac{1}{2}\zeta F_{\zeta\zeta} + \tau(\tau-1)F_{\zeta\tau} + \tau^2(F_\zeta F_{\zeta\tau} - F_\tau F_{\zeta\zeta}) = 0. \quad (5.1)$$

It is already known that $F \equiv 0$ for $0 < \tau < 1$, while linear theory states that

$$F \sim 4\pi^{-1}(\tau-1)^{\frac{1}{2}}\zeta^{-1}e^{-\frac{1}{4}\zeta^2} \quad \text{for} \quad 0 < \tau - 1 \ll 1, \quad \zeta \to \infty. \quad (5.2)$$

Now the linearization of (5.1) is invalid when $\xi \equiv (\tau-1)\zeta^4 e^{\frac{1}{4}\zeta^2}$ is positive and $O(1)$, which gives a non-uniform region as $\tau \to 1+$ when $\zeta \gg 1$; however, when (5.2) is substituted into (5.1), it is seen that the dominant terms cancel. This leads us to conjecture that (5.2) represents the leading term in the solution to the non-linear boundary-layer equations as $\xi \to 0+$. The conjecture is justified when it is proved (i) that (5.2) is the unique solution to (5.1) for $\xi = O(1)$ with the correct behaviour as $\xi \to \infty$, and (ii) that there is no further region of non-uniformity within which $\xi = O(1)$. (An infinite number of solutions to (5.1) with an essential singularity at $\tau = 1$ do exist, but all have a rapidly oscillating part that is physically unrealistic and must be rejected.) These conclusions imply that the discontinuity in the velocity perpendicular to the axis of the plate is a natural consequence of the boundary-layer assumption.

To conclude, we can state the error involved in the calculations of the preceeding sections when the plate is real with a finite length l.

The point $P(x, y)$ is taken to be in the wake with $x > 0$; the origin represents the trailing edge and the point $(-l, 0)$ the leading edge. Any influence of the leading edge will be transported by convection to the point P after the time $(l+x)/U$; at this time the solution will completely break down. However, there is the physical effect of diffusion parallel to the plate; this transmits the effect of the leading edge to P instantaneously. From (4.10) we can see that, for times less than $(l+x)/U$, the error involved in ignoring the existence of the leading edge is exponentially small as

$$\exp\left\{-\frac{(l+x-Ut)^2}{4\nu t}\right\},$$

which it is reasonable to neglect.

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